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Darboux transformations of the Jaynes–Cummings Hamiltonian

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Abstract

A detailed analysis of matrix Darboux transformations under the condition that the derivative of the superpotential be self-adjoint is given. As a consequence, a class of symmetries associated with Schrödinger matrix Hamiltonians is characterized. The applications are oriented towards the Jaynes–Cummings eigenvalue problem, so that exactly solvable 2×2 matrix Hamiltonians of the Jaynes–Cummings type are obtained as supersymmetric partners. It is also established that, due to nontrivial symmetries, the Jaynes–Cummings Hamiltonian is a quadratic function of a Dirac-type Hamiltonian.

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1. Introduction

Eigenvalue problems with matrix Hamiltonians play an essential role in many physical applications. For instance, in scattering theory of composite particles, they appear as multichannel Hamiltonians. In quantum optics they are used for describing multilevel atoms interacting with a quantized radiation field. In the simplest case of a two-level atom this is a 2×2 matrix Hamiltonian and if the rotating wave approximation is used the well-known exactly solvable version of the Jaynes–Cummings (JC) Hamiltonian is involved (see e.g. [1, 2]). Sukumar and Buck [3] presented a number of exactly solvable models of a two-level atom coupled to a single mode radiation field. Here we are addressing the question if this class of exactly solvable Hamiltonians can be enlarged with the help of the well-known supersymmetry approach (or in other words, by means of Darboux transformations).

While analysing this question we were aware that the usual multichannel supersymmetric quantum mechanics was almost useless for this purpose. The reason is that this procedure is developed under the additional condition that the superpotential matrix $W(x)$ be Hermitian [4] but, as we will show below, this is not just the case of the Jaynes–Cummings Hamiltonian.

In this respect Andrianov *et al* [5] already considered the case where this assumption could be replaced by a weaker demand, namely, the derivative of the matrix $W(x)$ be Hermitian, which is equivalent to $W(x) - W^+(x) = C$, where C is a skew-Hermitian constant matrix. However, they excluded the possibility for C to be proportional to the Pauli matrix σ_2 (thus making it impossible to apply their method to the Jaynes–Cummings Hamiltonian) and, moreover, their approach was developed in detail only for a particular case. So, the main aim of this paper is to analyse carefully the problem of Hermiticity of the potential difference produced by general Darboux transformations [6, 7] and then apply these results to the Jaynes–Cummings eigenvalue problem.

Another interesting point we would like to stress here is a new character of the factorization property that we observe when dealing properly with the Hermiticity problem. To show this feature, first we find the transformation operator L, defined according to Goncharenko and Veselov [6] as a differential intertwiner of two matrix Hamiltonians: $Lh_0 = h_1L$. Once it is known, we can find formally its adjoint L^+ and compose it with L which gives us $L^+L = h_0 + s_0$, where s_0, in contrast to the usual SUSY approach [4], is not proportional to the identity matrix but, in general, it is a nontrivial (matrix) operator. Since L^+ takes part in the adjoint intertwining relation, $L^+h_1 = h_0L^+$, the superposition L^+L is a symmetry operator for h_0 meaning that actually s_0 is a symmetry operator too. Such an appearance of symmetry operators was first noted in [5] and treated as a *hidden symmetry* exhibiting within supersymmetry. Unfortunately, because of the above-mentioned restriction imposed on C, the method of [5] cannot be applied to get the hidden symmetries for the Jaynes–Cummings Hamiltonian, so that in this respect we will make use of our more general set-up.

The paper is organized as follows. In the next section, to fix the notation, we briefly describe the solution of the multiphoton Jaynes–Cummings eigenvalue problem pointing out how one can find 'non-physical' solutions. Section 3 is devoted to adapting the known technique of matrix Darboux transformations to Jaynes–Cummings Hamiltonians. In section 4 we give a careful analysis of general matrix Darboux transformation leading to Hermitian potentials. In particular, we find that they allow for a new factorization scheme responsible for the appearance of non-trivial symmetry operators. We indicate a way of establishing a one-to-one correspondence between the spaces of solutions, and finally construct an integral transformation operator. Section 5 establishes the link between the results of the previous section and the Jaynes–Cummings eigenvalue problem. Also, in this section, we explicitly construct new exactly solvable matrix Hamiltonians of the Jaynes–Cummings type. Some conclusions are drawn in the last section.

2. Jaynes–Cummings eigenvalue problem

Our starting point is the matrix Hamiltonian

$$h^{(k)} = \begin{pmatrix} N + \alpha & \beta a^k \\ \beta^* (a^+)^k & N \end{pmatrix} \tag{1}$$

describing a multiphoton interaction between a two-level atom and a single-mode radiation field [2, 3]. Here a and a^+ are the standard harmonic oscillator creation and annihilation operators, $N = a^+a$ is the number operator, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$ are parameters. The positive integer k corresponds to the number of photons the atom exchanges with the field. For the most interesting particular case of the one photon exchange, $k = 1$, we will use the special notation $h_{\mathrm{JC}} := h^{(1)}$. In what follows we need not only the known 'physical' eigenfunctions of h_{JC} corresponding to its discrete spectrum eigenvalues but also those solutions of the Schrödinger

equation which do not belong to the Hilbert space. Therefore, below we give a short outline of how these solutions may be obtained.

Let ψ_n be the usual number operator eigenstates, $N\psi_n = n\psi_n$, with the properties $a^k\psi_n = (n+1-k)_k^{1/2}\psi_{n-k}$, $(a^+)^k\psi_n = (n+1)_k^{1/2}\psi_{n+k}$. Here $(x)_k$ denotes the standard Pochhammer symbol, $(x)_k = \Gamma(x+k)/\Gamma(x)$. Hamiltonian (1) can be diagonalized by a rotation inside any two-dimensional space spanned by the vectors $(\psi_n, 0)^t$ and $(0, \psi_{n+k})^t$ (the superscript 't' meaning the transposition). Indeed, if we write an eigenvector Ψ_E of $h^{(k)}$, $h^{(k)}\Psi_E = E\Psi_E$, in the form $\Psi_E = (c_1\psi_n, c_2\psi_{n+k})^t$, then the eigenvalue problem is reduced to a simple system of two linear homogeneous equations for the coefficients c_1 and c_2, which gives us the spectrum

$$E = E_n^{1,2} = n + \tfrac{1}{2}(k+\alpha) \pm \sqrt{\tfrac{1}{4}(\alpha-k)^2 + |\beta|^2(n+1)_k} \tag{2}$$

and the coefficients

$$c_2 = \frac{|\beta|}{\beta}\frac{E-n-\alpha}{[|\beta|^2(n+1)_k + (n+\alpha-E)^2]^{1/2}} \qquad c_1 = \sqrt{1-|c_2|^2}. \tag{3}$$

The set of eigenvectors thus determined, $\Psi_n = (c_1\psi_n, c_2\psi_{n+k})^t, n = 0, 1, \ldots$ is not complete in the Hilbert space but the missing ones can be readily obtained by noting that $\Psi_j^0 = (0, \psi_j)^t$ are also eigenvectors of $h^{(k)}$, corresponding to an equidistant part of the spectrum $h^{(k)}\Psi_j^0 = j\Psi_j^0$, $j = 0, \ldots, k-1$.

An interesting feature of Hamiltonian (1) is that for $k > 2$ the square root in (2) dominates for large n leading to the unbounded character of the spectrum both from above and from below. From this point of view this Hamiltonian is similar to a Dirac Hamiltonian in relativistic quantum mechanics.

The easiest way to get 'unphysical' solutions for $h^{(k)}$ is to note that in the above construction we needed just a couple of states $\{\psi_n, \psi_{n+k}\}$ related by the operators a^k, $(a^+)^k$. We can also take these vectors from the unphysical sector. For instance, choosing them among the basis vectors $\{\psi_{-n}\}_{n=1}^{\infty}$ of the (unphysical) upper bounded representation of the oscillator algebra where $a^+\psi_{-n} = \sqrt{-n+1}\,\psi_{-n+1}$, $a\psi_{-n} = \sqrt{-n}\psi_{-n-1}, a^+\psi_{-1} = 0$ we will get eigensolutions $\Psi = (c_1\psi_{-n}, c_2\psi_{-n+k})^t$ corresponding to an eigenvalue E. The expressions for E and c_1, c_2 are given in this case by the same formulae (2) and (3), where n has to be simply replaced by $-n$. In the coordinate representation where $\psi_n(x) = (\sqrt{\pi}n!2^n)^{-1/2}\exp(-x^2/2)H_n(x)$, $H_n(x)$ being the Hermite polynomials, the negative eigenvectors are realized as $\psi_{-n}(x) = \psi_{n-1}(\mathrm{i}x)$. Here the eigenvalues can be complex according to the sign under the square root in (2) that now may be negative depending on n, k, α and β. A particular set of nonphysical eigenvectors is given by $(\psi_{-j}, 0)^t$, $j = 1, \ldots, k-1$.

Even we can use a general solution $\xi_1(x)$ of the oscillator equation $a^+a\xi_1(x) = \varepsilon\xi_1(x)$ together with $\xi_2 = (a^+)^k\xi_1$, provided $\xi_1 \propto a^k\xi_2$, so that the nonphysical eigenfunction has the form $\Xi = (c_1\xi_1, c_2\xi_2)^t$. The coefficients c_1, c_2 depend now on ε and can be easily computed, but, in general, they are complex too.

3. Darboux transformation of the Jaynes–Cummings Hamiltonian

To construct the Darboux transformation for the Jaynes–Cummings Hamiltonian we are using the existing approach developed for the matrix Schrödinger equation [6, 7]. Therefore, we first reduce the Jaynes–Cummings eigenvalue problem to the Schrödinger equation with a matrix-valued potential. Then applying Darboux transformations we get new exactly solvable matrix potentials for the Schrödinger Hamiltonian and the use of the inverse transformation allows us to obtain the desired transformation for the Jaynes–Cummings Hamiltonian.

We start with rewriting $h_{\rm JC}$ in the coordinate representation as

$$h_0^{\rm JC} = -\partial_x^2 + V_0^{\rm JC} + b\gamma\partial_x \qquad \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad h_{\rm JC} := \tfrac{1}{2}h_0^{\rm JC} \tag{4}$$

with

$$V_0^{\rm JC} = \begin{pmatrix} x^2 - 1 + 2\alpha & bx \\ bx & x^2 - 1 \end{pmatrix} \qquad b = \sqrt{2}\beta \tag{5}$$

where, for simplicity, we have taken β real. A Hamiltonian of the form (4) without the second-order derivative term is referred to as a Dirac-like Hamiltonian, while the absence of the first-order derivative leads to a pure matrix Schrödinger Hamiltonian. As the operator $h_0^{\rm JC}$ contains the first-order derivative term, the method of [6, 7] cannot be applied to it directly. This undesirable term is easily removed by the unitary transformation

$$h_0 = \mathbb{U}^+ h_0^{\rm JC}\mathbb{U} \qquad \mathbb{U} = (\mathbb{U}^+)^{-1} = {\rm e}^{\frac{1}{2}b\gamma x} \tag{6}$$

which gives us the Schrödinger Hamiltonian

$$h_0 = -\partial_x^2 + V_0 \qquad V_0 = \mathbb{U}^+ V_0^{\rm JC}\mathbb{U} - \tfrac{1}{4}b^2. \tag{7}$$

The Darboux transformation operator L for a Schrödinger-type Hamiltonian is known to be in the form [6, 7]

$$L = \partial_x - W \qquad W = \mathcal{U}_x\mathcal{U}^{-1} \tag{8}$$

leading to the transformed potential

$$V_1 = V_0 - 2W_x \tag{9}$$

where the subscript x means the derivative with respect to x. The matrix $\mathcal{U}$ is a solution to the matrix eigenvalue problem

$$h_0\mathcal{U} = \mathcal{U}\Lambda \tag{10}$$

with the eigenvalue Λ being a diagonal matrix, $\Lambda = {\rm diag}(\lambda_1, \lambda_2)$. In this case λ_k are eigenvalues of h_0 corresponding to column vectors $U_k = (u_{1,k}, u_{2,k})^t$, $k = 1, 2$, and the matrix $\mathcal{U}$ is just composed of these columns, $\mathcal{U} = (U_1, U_2)$.

Once the Hamiltonian $h_1 = -\partial_x^2 + V_1$ is determined we realize the inverse transform to express all quantities in terms of the eigenvalue problem for $h_0^{\rm JC}$. For instance, $\mathcal{U}^{\rm JC} = \mathbb{U}\mathcal{U}$ will be an $h_0^{\rm JC}$ matrix eigensolution, preserving the same eigenvalue Λ,

$$h_0^{\rm JC}\mathcal{U}^{\rm JC} = \mathcal{U}^{\rm JC}\Lambda. \tag{11}$$

As for the potential $V_1^{\rm JC}$, defined by the expression $h_1^{\rm JC} = \mathbb{U}h_1\mathbb{U}^+ := -\partial_x^2 + V_1^{\rm JC} + b\gamma\partial_x$, we get

$$V_1^{\rm JC} = V_0^{\rm JC} - 2W_x^{\rm JC} + b(\gamma W^{\rm JC} - W^{\rm JC}\gamma). \tag{12}$$

It is easy to see that the operator L (8) is covariant under the transformation $\mathbb{U}$:

$$L^{\rm JC} = \mathbb{U}L\mathbb{U}^+ = \partial_x - W^{\rm JC}. \tag{13}$$

Keeping the usual SUSY terminology (see e.g. [4]) we will refer to the matrix $W^{\rm JC} := \mathcal{U}_x^{\rm JC}(\mathcal{U}^{\rm JC})^{-1}$ as a *Jaynes–Cummings superpotential*. It is interesting to note that the change in the transformed potential (12) has two contributions. The term $-2W_x^{\rm JC}$ in (12) corresponds to the usual expression for the potential difference due to the Darboux transformation (see e.g. [8, 9]). The next term proceeds from the first derivative member of the Hamiltonian $h_0^{\rm JC}$ and coincides with the potential difference for Darboux transformed Dirac systems (see [10]).

We see from (12) and (13) that the new potential and the transformation operator are defined in terms of a matrix-valued solution, $\mathcal{U}^{\rm JC}$, of the initial Hamiltonian. Therefore, as usual [9] we call it a *transformation function*. To keep the potential difference regular we have to impose an additional condition on $\mathcal{U}^{\rm JC}$: $\det\mathcal{U}^{\rm JC} \neq 0$. Another additional condition must be imposed on $\mathcal{U}^{\rm JC}$ if we require that $V_1^{\rm JC}$ be represented by a self-adjoint matrix. It is not difficult to see that if we restrict ourselves to real potentials, the last term in (12) is symmetric for any matrix $W^{\rm JC}$. Therefore, the potential $V_1^{\rm JC}$ will be self-adjoint if $W^{\rm JC}$ is real and its derivative symmetric, i.e., $W^{\rm JC} - (W^{\rm JC})^+ = C^{\rm JC}$ where $C^{\rm JC} = -(C^{\rm JC})^+$ is a skew Hermitian constant which because of the real character of $W^{\rm JC}$ should be proportional to γ, $C^{\rm JC} = c_0\gamma$. Now using the relation $W^{\rm JC} = \mathbb{U}W\mathbb{U}^+ + \frac{b}{2}\gamma$ we find

$$W - W^+ = c\gamma \qquad c = c_0 - b. \tag{14}$$

We note that both superpotentials $W^{\rm JC}$ and W can never be simultaneously Hermitian. In section 5 we illustrate that both possibilities (the Hermiticity of $W^{\rm JC}$ or that of W) may take place. From here it follows that the usual approach of the multichannel supersymmetric quantum mechanics based on Hermitian superpotential matrices, $W = W^+$, cannot be applied to the Hamiltonian (6). The above condition should be replaced by a weaker demand (14).

So, we have established how the matrix Darboux transformation should be modified when applied to the Jaynes–Cummings Hamiltonian. It can be considered as a special case of the general transformation introduced by Goncharenko and Veselov [6] with an additional condition of type (14). Now we are going to investigate the implications this restriction imposes in general and later we will apply them to the particular case of the Jaynes–Cummings Hamiltonian.

4. Properties of the Darboux transformation for the matrix Schrödinger equation

Darboux transformation of the usual (one-component) Schrödinger equation has a number of nice properties (see e.g. [9]), making it popular in different fields of theoretical and mathematical physics. In this section we show how some of them are translated to the matrix level. In our opinion the most interesting properties are the following:

- The most convenient way to introduce the transformation operator associated with the name Darboux is to define it as a *differential intertwiner* between two Hamiltonians h_0 and h_1. Defining (formally) an adjoint operation such that the Hamiltonians are self-adjoint one gets the adjoint intertwining relation

$$Lh_0 = h_1L \qquad L^+h_1 = h_0L^+. \tag{15}$$

- Given L and L^+ one can compose them thus discovering *factorization properties* first discussed by Schrödinger and studied in detail by Infeld and Hull.
- In matrix notation the above properties can equivalently be rewritten as commutation and anticommutation relations giving rise to a *supersymmetry algebra* commonly known also as *supersymmetric quantum mechanics*.
- There exists a procedure allowing one to realize a *one-to-one correspondence* between the spaces of solutions of the Hamiltonians h_0 and h_1. Therefore, if h_0 is exactly solvable, then h_1 is solvable as well. In this case it is rather easy to find the spectrum of h_1 when the operators are defined in a Hilbert space. In particular, one can conclude that the spectrum of h_1 coincides with the spectrum of h_0 with the possible exception of a finite number of levels. In the simplest case either the spectra may differ only by the ground-state level (so called case of the 'exact supersymmetry') or they may be identical ('broken supersymmetry').

- Making use of a special two-step differential transformation one recovers an *integral transformation* (sometimes called Luban–Pursey method) leading, in general, to a relation between the method of Darboux transformations and inverse scattering method [12].
- There exist nice Crum–Krein formulae for a compact representation of the resulting action of a chain of first-order transformations. Recently they have been generalized to the matrix case [7], so we will not discuss this point here.

4.1. Factorization properties and underlying supersymmetry

First we will study how the factorization properties are translated to the matrix level.

Since

$$W^+ = W - C \qquad W = \mathcal{U}_x\mathcal{U}^{-1} \tag{16}$$

$$h_0\mathcal{U} = \mathcal{U}\Lambda \tag{17}$$

we have $L^+ = -\partial_x - W + C$ which results in the following superposition:

$$L^+L = h_0 + s_0 \qquad s_0 = CL - \mathcal{U}\Lambda\mathcal{U}^{-1}. \tag{18}$$

To find the superposition LL^+ we express L in terms of W^+: $L = \partial_x - W^+ - C$ which yields

$$LL^+ = h_1 + s_1 \qquad s_1 = -CL^+ - (\mathcal{U}\Lambda\mathcal{U}^{-1})^+. \tag{19}$$

From here it follows that $s_0 = s_0^+$ commutes with h_0 and $s_1 = s_1^+$ commutes with h_1 and they are intertwined by L, $Ls_0 = s_1L$, (these facts can also be checked through a direct calculation). Nevertheless, it is necessary to stress here that this observation is correct only formally. When all operators are considered as acting in a Hilbert space they may not be commuting because of different domains of definitions and the subject should be studied more carefully.

From (17) and its adjoint form another nice property of the transformation function $\mathcal{U}$ follows:

$$2(\mathcal{U}\Lambda\mathcal{U}^{-1})_x = CV_1 - V_0C = Ch_1 - h_0C$$

which shows that for the particular case $C = 0$ the matrix $\mathcal{U}\Lambda\mathcal{U}^{-1}$ is constant. We would also like to stress that even in this particular case the factorizations (18) and (19) do not coincide with those giving rise to the usual multichannel supersymmetric quantum mechanics: $L^+L = h_0 - \lambda I$ and $LL^+ = h_1 - \lambda I$ where λ is known as factorization constant and I is the unit matrix. This fact was already mentioned in [7]. Another point worth noting is that in contrast with the paper [5], where similar symmetry operators have been found, we give their explicit expression in terms of a solution $\mathcal{U}$ of the initial eigenvalue problem. In fact, such nontrivial symmetries may exist only if the Hamiltonian h_0 has at least one matrix solution $\mathcal{U}$ with property (16).

Another interesting observation is that if the spectrum of h_0 is nondegenerate and essentially self-adjoint operators h_0 and s_0 have a common set (dense in the Hilbert space) in their domains of definition, where they commute, they should be related by a functional dependence. This statement follows from the fact that if the spectrum of h_0 is nondegenerate then a complete set of operators commuting with h_0 consists of only one operator which is h_0 itself. Hence, by definition of a complete set of operators (see e.g. [11]) any other self-adjoint operator commuting with h_0 is a function of h_0. In this case all eigenfunctions of h_0 can be found by solving the eigenvalue problem for s_0, which is a first-order differential operator. We shall see later that just the Jaynes–Cummings Hamiltonian presents a nontrivial example of such a situation.

From the point of view of supersymmetric quantum mechanics we can construct here a wider (extended) superalgebra than that usually appearing in the one-component case. In addition to the super-Hamiltonian

$$H = \begin{pmatrix} h_0 & 0 \\ 0 & h_1 \end{pmatrix}$$

and mutually conjugated nilpotent supercharges

$$Q = (Q^+)^+ = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}$$

we have the symmetry operator

$$S = \begin{pmatrix} s_0 & 0 \\ 0 & s_1 \end{pmatrix}.$$

Now, as usual, the intertwining relations (15) are equivalent to the commutation of H and S with the supercharges and the factorization properties are translated to the anticommutation relation for the supercharges $QQ^+ + Q^+Q = H + S$. It is also evident that H and S commute between them. According to the remark made in the previous paragraph, these relations are also rather formal and when a concrete Hilbert space is considered it is necessary to take care of domains where the operators act.

4.2. One-to-one correspondence between the spaces of solutions

Being differential, the operators L and L^+ have nontrival kernels. Therefore, the correspondence between the spaces of solutions of the Schrödinger equations for h_0 and h_1 should be studied carefully. More precisely, a detailed analysis is necessary for kernel spaces of these operators which are eigenfunctions of h_0 and h_1 corresponding to the eigenvalue matrix Λ. In the one-component case $\mathrm{Ker}(h_0-\lambda)$ is a two-dimensional space easily determined by a solution u, $h_0u = \lambda u$. This possibility is based on the property stating that the Wronskian of two solutions corresponding to the same eigenvalue must be a constant. First we are going to show how this property translates to the matrix case and then how it helps us to establish a one-to one correspondence.

The conjugation equation for the superpotential (16) being rewritten in another form

$$W(\mathcal{U},\mathcal{U}) := \mathcal{U}_x^+\mathcal{U} - \mathcal{U}^+\mathcal{U}_x + \mathcal{U}^+C\mathcal{U} = 0 \tag{20}$$

establishes a property of the transformation function $\mathcal{U}$. We relate it to the fact that the Wronskian of a solution with itself vanishes. Therefore we call the quantity $W(\mathcal{U},\mathcal{U})$, defined by the middle part of (20), Wronskian with coinciding arguments. Quite naturally for different arguments it is defined as

$$W(\mathcal{V},\mathcal{U}) := \mathcal{V}_x^+\mathcal{U} - \mathcal{V}^+\mathcal{U}_x + \mathcal{V}^+C\mathcal{U}. \tag{21}$$

We note that for $C = 0$ and $\mathcal{U}$, $\mathcal{V}$ being the usual functions (not matrices) it coincides with the usual expression for the Wronskian. Therefore, we consider (21) as a matrix generalization of the Wronskian.

Now we will show that if $\mathcal{U}$ is a solution to (17), satisfying (16), then there exists another solution to the same equation, $\mathcal{V}$, which can be found from the condition $W(\mathcal{V},\mathcal{U}) = -1$.

We proceed first to find an explicit expression for $\mathcal{V}$. Multiplying the equation $W(\mathcal{V},\mathcal{U}) = -1$ by $\mathcal{U}^{-1}$ from the right and using (16) we get

$$\mathcal{V}_x^+ - \mathcal{V}^+(\mathcal{U}^+)^{-1}\mathcal{U}_x^+ = \mathcal{U}^{-1}. \tag{22}$$

Multiplying the adjoint form of (22) by $\mathcal{U}^{-1}$ from the left we obtain $(\mathcal{U}^{-1}\mathcal{V})_x = \mathcal{U}^{-1}(\mathcal{U}^+)^{-1}$, which gives us the final result

$$\mathcal{V} = \mathcal{U}\int_{x_0}^{x}(\mathcal{U}^+\mathcal{U})^{-1}\,\mathrm{d}x + \mathcal{U}C_1. \tag{23}$$

In general, we cannot neglect the matrix integration constant C_1. This is due to the evident property of (17) not to be a linear equation with respect to the multiplication on matrices. Only if this constant takes a definite value, is this function a solution to (17).

Now we proceed to find C_1. For this purpose we put the function (23) into equation (17). Using the same equation for $\mathcal{U}$ and (16) we obtain

$$V_0\mathcal{U}C_1 - C_1(\mathcal{U}^+)^{-1} - \mathcal{U}_{xx}C_1 = \mathcal{U}\int_{x_0}^{x}(\mathcal{U}^+\mathcal{U})^{-1}\,\mathrm{d}x\Lambda - \mathcal{U}\Lambda\int_{x_0}^{x}(\mathcal{U}^+\mathcal{U})^{-1}\,\mathrm{d}x + \mathcal{U}C_1\Lambda. \tag{24}$$

Here it is necessary to calculate the commutator of Λ with the integral on the right-hand side of (24), but first we have to find the commutator of Λ with $(\mathcal{U}^+\mathcal{U})^{-1}$.

From (16) and (17) it is easy to get

$$\Lambda\mathcal{U}^+\mathcal{U} - \mathcal{U}^+\mathcal{U}\Lambda = (\mathcal{U}^+C\mathcal{U})_x$$

which can readily be transformed into

$$\Lambda(\mathcal{U}^+\mathcal{U})^{-1} - (\mathcal{U}^+\mathcal{U})^{-1}\Lambda = (\mathcal{U}^{-1}C(\mathcal{U}^+)^{-1})_x$$

which gives us

$$\int_{x_0}^{x}(\mathcal{U}^+\mathcal{U})^{-1}\,\mathrm{d}x\Lambda - \Lambda\int_{x_0}^{x}(\mathcal{U}^+\mathcal{U})^{-1}\,\mathrm{d}x = -\mathcal{U}^{-1}C(\mathcal{U}^+)^{-1} + C_2.$$

Here

$$C_2 = (\mathcal{U}^{-1}C(\mathcal{U}^+)^{-1})_{x=x_0} = -C_2^+. \tag{25}$$

Now once again using (17) we obtain from (24) the equation for C_1:

$$C_2 + C_1\Lambda - \Lambda C_1 = 0 \qquad C_1^+ = C_1. \tag{26}$$

Since we just reduced the Schrödinger equation for $\mathcal{V}$ to the last equation for C_1, this means that with the constant C_1 thus determined, the function $\mathcal{V}$ (23) satisfies the Schrödinger equation (17).

Here we would like to remark that L is a bijective operator from the n-dimensional vector space $\mathrm{Ker}\,(h_0 - E)$ onto $\mathrm{Ker}\,(h_1 - E)$, for any $E \neq \lambda_k$, $k = 1, 2, \ldots, n$ (recall that λ_k corresponds to the column eigenvectors U_k in the matrix $\mathcal{U}$), while L^+ realizes this isomorphism in the opposite way. As $L\mathcal{U} = 0$, to find the solutions of the Schrödinger equation for h_1 with the eigenvalues λ_k, we act by L on the second matrix solution $\mathcal{V}$, which yields

$$\overline{\mathcal{U}} = L\mathcal{V} = (\mathcal{U}^+)^{-1} \qquad h_1\overline{\mathcal{U}} = \overline{\mathcal{U}}\Lambda. \tag{27}$$

Since Λ is supposed to be diagonal, the columns of $\overline{\mathcal{U}}$ are eigenvectors of h_1 with eigenvalues λ_k.

An obvious but necessary remark is that $L^+\overline{\mathcal{U}} = 0$. Moreover, it is easy to see that equation (16) is covariant under the transformation $\mathcal{U} \to \overline{\mathcal{U}} = (\mathcal{U}^+)^{-1}$ meaning that $\overline{\mathcal{U}}$ has the same property. Hence, another solution of the eigenvalue problem for h_1 corresponding to the matrix eigenvalue Λ can be found using the same formula (23) applied this time to $\overline{\mathcal{U}}$:

$$\overline{\mathcal{V}} = (\mathcal{U}^+)^{-1}\left(\int_{x_0}^{x}\mathcal{U}^+\mathcal{U}\,\mathrm{d}x + \overline{C}_1\right) \qquad h_1\overline{\mathcal{V}} = \overline{\mathcal{V}}\Lambda. \tag{28}$$

Once again because of the diagonal character of Λ, columns of $\overline{\mathcal{V}}$ are eigenvectors of h_1 with eigenvalues λ_k. The constant $\overline{C}_1$ in (28) should be determined, in a similar way as C_1, from (23).

In summary, L and L^+ realize a one-to-one correspondence between the spaces of solutions for all $E \neq \lambda_k, k = 1, \ldots, n$. For $E = \lambda_k$ this correspondence may be established by hand: $\mathcal{U} \leftrightarrow \overline{\mathcal{U}}$, $\mathcal{V} \leftrightarrow \overline{\mathcal{V}}$, and extended by linearity.

4.3. Integral transformations

Let a matrix constant C_{01} be such that $(\mathcal{U}^+)^{-1}C_{01}$ is the solution to equation (27). Then the function $\widetilde{\mathcal{V}}_1 = \overline{\mathcal{V}} + (\mathcal{U}^+)^{-1}C_{01}$ with $\overline{\mathcal{V}}$ as given in (28) is the solution to the same equation and can be taken as the transformation function for the next transformation step. After some algebra one gets

$$(\widetilde{\mathcal{V}}_1)_x(\widetilde{\mathcal{V}}_1)^{-1} = -W + W_2 \qquad W_2 := \mathcal{U}\left[\int_{x_0}^{x} \mathcal{U}^+\mathcal{U}\,\mathrm{d}x + C_0\right]^{-1}\mathcal{U}^+ \tag{29}$$

where $C_0 = C_{01} + \overline{C}_1$. So the potential $V_2 = V_1 - 2((\widetilde{\mathcal{V}}_1)_x(\widetilde{\mathcal{V}}_1)^{-1})_x$ is given by

$$V_2 = V_0 + \Delta V_2 \qquad \Delta V_2 = -2(W_2)_x. \tag{30}$$

The potential difference ΔV_2 from (30) and (29) is self-adjoint for any matrix $\mathcal{U}$. Nevertheless, we can use here only those $\mathcal{U}$ which give rise to a self-adjoint derivative of the superpotential, $W_x^+ = W_x$, since only under condition (16) is function (28) a solution to the Schrödinger equation after the first transformation step. An advantage of this formula with respect to (9) could be a much easier possibility to get an everywhere nonsingular resulting potential since just the constant C_0 can be used for this purpose whereas in (9) we do not have any freedom of this kind. Moreover, in such a way one can get families of isospectral and isophase (known also as phase-equivalent) potentials. Applying twice the appropriately changed formula (8) we express solutions of the Schrödinger equation with the potential (30) in terms of solutions of the initial eigenvalue problem

$$\begin{aligned} &\varphi_E = \mathcal{U}\Lambda\mathcal{U}^{-1}\psi_E - \psi_E E - (C + W_2)(\psi_{Ex} - \mathcal{U}_x\mathcal{U}^{-1}\psi_E) \\ &h_2\varphi_E = \varphi_E E \qquad h_2 = -\partial_x^2 + V_2. \end{aligned} \tag{31}$$

Here ψ may be both vector-valued and matrix-valued eigenfunctions of h_0 with E being a number in the first case and a diagonal matrix in the second case, $h_0\psi_E = \psi_E E$. As usual, for $E = \Lambda$ and $\psi_E = \mathcal{U}$, the right-hand side of (31) vanishes, but the missing eigensolutions correspond to $\left(\widetilde{\mathcal{V}}_1^+\right)^{-1}$.

5. SUSY partners for the Jaynes–Cummings Hamiltonian

5.1. Properties of the Darboux transformation for the Jaynes–Cummings Hamiltonian

Using the relation between the Jaynes–Cummings Hamiltonian and the Schrödinger Hamiltonian given by equation (6), we can readily apply the results of the previous section to the Jaynes–Cummings eigenvalue problem.

First we note that according to (14) even for a self-adjoint Jaynes–Cummings superpotential, the constant C introduced in the previous section is not zero but $C = -b\gamma$ (see (14)). The factorization properties (18) and (19) together with the symmetry operators are covariant under this transformation. Therefore the symmetry operator for the Jaynes–Cummings Hamiltonian can be found from a diagonal superpotential matrix. Since $\mathbb{U}$

commutes with $C = -b\gamma$ formula (25) for the constant C_2 is covariant also. The unitary transformation (6) does not affect the eigenvalue matrix Λ^{JC}. Therefore equation (26) for the constant C_1 remains intact. Equations (27) and (28) for solutions of the transformed equations with the eigenvalue Λ^{JC} are also covariant, but now they are valid for the self-adjoint Jaynes–Cummings superpotential W^{JC}. For the integral transformation the transformed potential is given by the same formula (12), where W^{JC} is replaced by W_2^{JC}, which is given by (29), where all quantities are related now with the Jaynes–Cummings system. Formula (31) for solutions of the equation with this potential is also covariant.

5.2. Examples

1. In the simplest case the transformation function can be taken in the form (see section 2)

$$\mathcal{U}^{\mathrm{JC}} = \begin{pmatrix} 0 & \mathrm{e}^{x^2/2} \\ \mathrm{e}^{-x^2/2} & 0 \end{pmatrix}. \tag{32}$$

It corresponds to the diagonal eigenvalue matrix $\Lambda = \mathrm{diag}(0, 2\alpha - 2)$ and diagonal superpotential $W^{\mathrm{JC}} = \mathrm{diag}(x, -x)$. Formula (12) gives us the transformed potential

$$V_1^{\mathrm{JC}} = \begin{pmatrix} x^2 + 2\alpha - 3 & -bx \\ -bx & x^2 + 1 \end{pmatrix} \tag{33}$$

and finally the transformed Hamiltonian

$$h_1^{\mathrm{JC}} = 2\begin{pmatrix} a^+a + \alpha - 2 & -ba^+ \\ -ba & a^+a \end{pmatrix}. \tag{34}$$

Solutions of the Schrödinger equation with this Hamiltonian can be obtained from the solutions of the Jaynes–Cummings eigenvalue problem using the transformation operator

$$L^{\mathrm{JC}} = \sqrt{2}\begin{pmatrix} a^+ & 0 \\ 0 & a \end{pmatrix}. \tag{35}$$

Using (18) we find the symmetry operator

$$s_0 = b\gamma\partial_x + \begin{pmatrix} 2\alpha + 2 & bx \\ bx & 0 \end{pmatrix} = 2\begin{pmatrix} \alpha + 1 & ba \\ ba^+ & 0 \end{pmatrix}. \tag{36}$$

The spectrum of the Jaynes–Cummings Hamiltonian is nondegenerate and the set of all finite linear combinations of its eigenfunctions is dense in the corresponding Hilbert space. The operator s_0 (36) is well defined on this set where it commutes with h_{JC}. Hence, as expected, the latter is a second-order polynomial of s_0: $b^2 h_0^{\mathrm{JC}} = (s_0 - 4)(s_0 + b^2 - 2\alpha - 2)$.

2. Another possibility is to construct the transformation function $\mathcal{U}^{\mathrm{JC}}$ from two vectors of the type $(c_1\psi_n, c_2\psi_{n+1})^t$ with the coefficients $c_{1,2}$ given in (3) corresponding to the energies $E_n^{1,2}$ (2) at $k = 1$. For $n = 0$ we easily get the following superpotential:

$$W^{\mathrm{JC}} = \begin{pmatrix} -x & 0 \\ 0 & \frac{1}{x} - x \end{pmatrix}. \tag{37}$$

We note that it is singular at the origin and cannot give rise to a self-adjoint potential for the spectral problem on the whole real line. Nevertheless, up to a constant shifting it produces the same symmetry operator (36). Moreover, since at any n both even and odd Hermite polynomials are involved in this construction, the superpotentials we can obtain in this way are singular at the origin for any n.

3. As usually happens in the method of Darboux transformations (see e.g. [9]), a singular point can be removed by the next transformation if it is realized with the help of a transformation

function corresponding to the adjacent spectral point. Moreover, two consecutive first-order transformations can be replaced by a single second-order transformation (for details see [7]). As a result we get the following potential difference:

$$\Delta V = \frac{4}{(1+2x^2)^2}\begin{pmatrix}(1+2x^2)^2 & bx(1+2x^2)\\ bx(1+2x^2) & 4(x^4+8x^2-1)\end{pmatrix}. \tag{38}$$

We note here that the spectrum of the Hamiltonian $h_2^{\mathrm{JC}} = h_0^{\mathrm{JC}} + \Delta V$ coincides with the spectrum of h_0^{JC} except for four levels $E_{0,1}^{1,2}$ which are removed by this transformation. It is also clear that by taking other values of n one can remove any four levels $E_{n,n+1}^{1,2}$. For this purpose we are using two transformation functions

$$\mathcal{U}_1^{\mathrm{JC}} = \begin{pmatrix}\psi_n(x) & -A_1\psi_n(x)\\ A_1\psi_{n+1}(x) & \psi_{n+1}(x)\end{pmatrix} \qquad \mathcal{U}_2^{\mathrm{JC}} = \begin{pmatrix}\psi_{n+1}(x) & -A_2\psi_{n+1}(x)\\ A_2\psi_{n+2}(x) & \psi_{n+2}(x)\end{pmatrix} \tag{39}$$

and the algorithm developed in [7]. After some algebra we find that the second-order superpotential is diagonal, $W_2^{\mathrm{JC}} = \mathrm{diag}(w_0, w_1)$, with the entries $w_k = 2\psi_{n+k}\psi_{n+k+1}/(\psi'_{n+k}\psi_{n+k+1} - \psi_{n+k}\psi'_{n+k+1})$, $k = 0, 1$. Here in the denominator we see the Wronskian of two consecutive discrete spectrum eigenfunctions of the usual harmonic oscillator potential which is known to be nodeless [9]. Therefore it gives 'good' potential differences for any non-negative integer n. For instance, the case $n = 1$ corresponds to the following potential difference:

$$\Delta V = 4\begin{pmatrix}\frac{4x^4+8x^2-1}{(1+2x^2)^2} & b\left(\frac{4x^2}{3+4x^4} - \frac{x}{1+2x^2}\right)\\ b\left(\frac{4x^2}{3+4x^4} - \frac{x}{1+2x^2}\right) & 1+\frac{8x^2(4x^4-9)}{(3+4x^4)^2}\end{pmatrix}. \tag{40}$$

4. The creation of new energy levels is also possible. For this purpose we need solutions of the Schrödinger equation, which do not belong to the Hilbert space discussed in section 2. They are constructed with the help of the functions $\varphi_n(x) = \psi_n(\mathrm{i}x)$. For the transformation function

$$\mathcal{U}^{\mathrm{JC}} = \begin{pmatrix}\varphi_0 & A\varphi_{n+1}\\ 0 & \varphi_n\end{pmatrix} \qquad n = 0, 1, \ldots \tag{41}$$

the superpotential reads

$$W^{\mathrm{JC}} = \begin{pmatrix}\frac{\varphi'_0(x)}{\varphi_0(x)} & A\frac{\varphi_0(x)\varphi'_{n+1}(x) - \varphi'_0(x)\varphi_{n+1}(x)}{\varphi_0(x)\varphi_n(x)}\\ 0 & \frac{\varphi'_n(x)}{\varphi_n(x)}\end{pmatrix}. \tag{42}$$

Using the recurrence relation for the Hermite polynomial we find that its non-zero off-diagonal element does not depend on x, thus giving us a nontrivial example of a non-Hermitian up to a constant superpotential. Substituting the logarithmic derivatives by their expressions we get its final form

$$W^{\mathrm{JC}} = \begin{pmatrix}x & A\sqrt{2n+2}\\ 0 & x + \frac{2niH_{n-1}(\mathrm{i}x)}{H_n(\mathrm{i}x)}\end{pmatrix}. \tag{43}$$

We note that for even n the Hermite polynomials $H_n(\mathrm{i}x)$ are nodeless whereas $H_{n-1}(\mathrm{i}x)$ are purely imaginary. So, for $n = 0, 2, \ldots$ this formula gives us 'good' potential differences provided A is real which is not always the case. For instance, for $n = 0$ the potential difference is a constant diagonal matrix $\Delta V = \mathrm{diag}(B_0, -4 - B_0)$, where $B_0 = -1 - \alpha \pm \sqrt{(\alpha-1)^2 - 2b^2}$. Two possible signs here and below are related to different signs in (2). The first nontrivial case corresponds to $n = 2$:

$$\Delta V = \begin{pmatrix}B_1 & \frac{4bx}{1+2x^2}\\ \frac{4bx}{1+2x^2} & -4\frac{4x^4+3}{(1+2x^2)^2} - B_1\end{pmatrix} \tag{44}$$

where $B_1 = -1 - \alpha \pm \sqrt{(\alpha-1)^2 - 6b^2}$. We have to note that after being shifted by a constant diagonal matrix this potential difference may be reduced to the previous example (38) at $n = 0$. Hence, to get an essentially new potential difference we have to consider $n = 4$:

$$\Delta V = \begin{pmatrix} B_2 & \frac{8bx(3+2x^2)}{4x^4+12x^2+3} \\ \frac{8bx(3+2x^2)}{4x^4+12x^2+3} & -4\frac{16x^8+64x^6+120x^4+45}{(4x^4+12x^2+3)^2} - B_2 \end{pmatrix} \tag{45}$$

where $B_2 = -1 - \alpha \pm \sqrt{(\alpha-1)^2 - 10b^2}$.

5. As the final example we give a one-parameter family of potentials isospectral with the Jaynes–Cummings Hamiltonian obtained with the help of the integral transformation (29)–(30):

$$\Delta V = 8\begin{pmatrix} \frac{1+xF(x)}{F^2(x)} & -\frac{b}{4}\left(\frac{1}{F(x)} + \frac{2x^2}{2x-F(x)}\right) \\ -\frac{b}{4}\left(\frac{1}{F(x)} + \frac{2x^2}{2x-F(x)}\right) & 2x\frac{2x+(x^2-1)F(x)}{(2x-F(x))^2} \end{pmatrix} \tag{46}$$

where $F(x) = \sqrt{\pi}\,\mathrm{e}^{x^2}(c + \mathrm{erf}(x))$ and real c is such that $|c| > 1$.

6. Conclusion

A careful analysis of the matrix Darboux transformation method has permitted us to establish such properties as: (i) a new factorization scheme, which is responsible for the appearance of an extended supersymmetry underlying matrix Hamiltonians and hidden symmetry operators, (ii) a one-to-one correspondence between the spaces of solutions, which allows us to readily determine the changes in the spectrum, and (iii) the integral transformation formula, which gives families of isospectral Hamiltonians. We applied these results to get a new symmetry operator for the Jaynes–Cummings Hamiltonian and built up many of its exactly solvable partners. A future step of this research will be the investigation of physical phenomena, such as for instance the collapse and revival of new Hamiltonians.

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